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Towards extending the Ahlswede–Khachatrian theorem to cross t-intersecting families

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ABSTRACT

Ahlswede and Khachatrian's diametric theorem is a weighted version of their complete intersection theorem, which is itself a well known extension of the t-intersecting Erdős–Ko–Rado theorem. The complete intersection theorem says that the maximum size of a family of subsets of $[n] = \{1, \ldots, n\}$, every pair of which intersects in at least t elements, is the size of certain trivially intersecting families proposed by Frankl. We address a cross intersecting version of their diametric theorem.

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $[n]$ are *cross t-intersecting* if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, A and B intersect in at least t elements. The p-weight of a k element subset A of $[n]$ is $p^k(1-p)^{n-k}$, and the weight of a family $\mathcal{A}$ is the sum of the weights of its sets. The weight of a pair of families is the product of the weights of the families.

The maximum p-weight of a t-intersecting family depends on the value of p. Ahlswede and Khachatrian showed that for p in the range $[\frac{r}{t+2r-1}, \frac{r+1}{t+2r+1}]$, the maximum p-weight of a t-intersecting family is that of the family $\mathcal{F}_r^t$ consisting of all subsets of $[n]$ containing at least $t+r$ elements of the set $[t+2r]$.

In a previous paper we showed a cross t-intersecting version of this for large t in the case that $r = 0$. In this paper, we do the same in the case that $r = 1$. We show that for p in the range $[\frac{1}{t+1}, \frac{2}{t+3}]$ the maximum p-weight of a cross t-intersecting pair of families, for $t \geq 200$, is achieved when both families are $\mathcal{F}_1^t$. Further, we show that except at the endpoints of this range, this is, up to isomorphism, the only pair of t-intersecting families achieving this weight.

1. Introduction

Let $[n] := \{1, 2, \ldots, n\}$ and let $\binom{[n]}{k}$ be the family of all k-subsets of $[n]$. For a positive integer t, the family $\mathcal{A} \subset 2^{[n]}$ is called *t-intersecting* if, for each $A, A' \in \mathcal{A}$, we have $|A \cap A'| \geq t$. Erdős, Ko, and Rado proved in [4] that, for each k and t, there exists $n_0 = n_0(k, t)$ such that if $n \geq n_0$ and a family of k-element subsets $\mathcal{A} \subset \binom{[n]}{k}$ is t-intersecting, then $|\mathcal{A}| \leq \binom{n-t}{k-t}$ with equality holding if and only if there is some $T \in \binom{[n]}{t}$ such that $\mathcal{A} = \{A \in \binom{n}{k} : T \subset A\}$. The exact bound $n_0(k, t) = (t+1)(k-t+1)$ was established by Frankl [5], where he introduced the random walk method, and independently by Wilson [9], where he used a linear programming bound due to Delsarte.

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Frankl also considered the case when $n < (t+1)(k-t+1)$. He defined t-intersecting families $\mathcal{F}_i^t$ by

$$\mathcal{F}_i^t = \mathcal{F}_i^t(n) = \left\{F \subset [n] : |F \cap [t+2i]| \geq t+i\right\},$$

and conjectured that if $\mathcal{A} \subset \binom{[n]}{k}$ is t-intersecting, then

$$|\mathcal{A}| \leq \max_i |\mathcal{F}_i^t \cap \binom{[n]}{k}|.$$

This conjecture was partially proved by Frankl and Füredi in [6], and was finally settled by Ahlswede and Khachatrian in the affirmative in [1] and [3]. This result, now known as the complete intersection theorem, is one of the most important results in extremal set theory.

Ahlswede and Khachatrian also obtained the p-weight version of their complete intersection theorem in [2]. This result, which they called the diametric theorem, applies to non-uniform families of subsets of $[n]$. To state the result, we let p be a real number with $0 < p < 1$, and let $q := 1-p$. For a family $\mathcal{F} \subset 2^{[n]}$, the *$p$-weight of $\mathcal{F}$* is defined by

$$\mu_p(\mathcal{F}) := \sum_{F \in \mathcal{F}} p^{|F|} q^{n-|F|}.$$

Ahlswede and Khachatrian showed that for $p \leq 1/2$ if $\mathcal{F} \subset 2^{[n]}$ is t-intersecting, then

$$\mu_p(\mathcal{F}) \leq \max_i \mu_p(\mathcal{F}_i^t). \tag{1}$$

Comparing $\mu_p(\mathcal{F}_i^t)$ and $\mu_p(\mathcal{F}_{i+1}^t)$, it can be shown that $\max_i \mu_p(\mathcal{F}_i^t) = \mu_p(\mathcal{F}_r^t)$ if and only if

$$\frac{r}{t+2r-1} \leq p \leq \frac{r+1}{t+2r+1}. \tag{2}$$

All values of $p \in (0, 1/2)$ fall into this range for some r, larger p yields larger r.

For a positive integer t, the families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are called *cross t-intersecting* if, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $|A \cap B| \geq t$. We consider an extension of (1) to cross t-intersecting families.

Conjecture 1. *If $\mathcal{A} \subset 2^{[n]}$ and $\mathcal{B} \subset 2^{[n]}$ are cross t-intersecting, then where r is such that p satisfies (2),*

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq \mu_p(\mathcal{F}_r^t)^2.$$

With Frankl, in [7], we verified the $r = 0$ case of the above conjecture for $t \geq 14$. In this paper we verify the $r = 1$ case of the conjecture for $t \geq 200$. This result is perhaps the first result concerning cross intersecting families, where optimal structures are different from the so-called trivial structure $\mathcal{F}_0^t$. To state our main result we need one more definition. Two families $\mathcal{G}_1, \mathcal{G}_2 \in 2^{[n]}$ are *isomorphic*, denoted by $\mathcal{G}_1 \cong \mathcal{G}_2$, if there is a permutation σ on $[n]$ such that $\mathcal{G}_1 = \left\{\{\sigma(k) : k \in G\} \ : \ G \in \mathcal{G}_2\right\}$.

Theorem 1. *Let n and t be integers with $n \geq t \geq 200$, and let p be such that $\frac{1}{t+1} \leq p \leq \frac{2}{t+3}$. If $\mathcal{A} \subset 2^{[n]}$ and $\mathcal{B} \subset 2^{[n]}$ are cross t-intersecting, then*

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq \left(\mu_p(\mathcal{F}_1^t)\right)^2 = \left((t+2)p^{t+1}q + p^{t+2}\right)^2. \tag{3}$$

Moreover, equality holds if and only if one of the following holds:

1. $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_0^t$ *and* $p = \frac{1}{t+1}$,
2. $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_1^t$ *and* $\frac{1}{t+1} \leq p \leq \frac{2}{t+3}$,
3. $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_2^t$ *and* $p = \frac{2}{t+3}$.

Remark that we do not attempt to optimize the range of t. The parts requiring t to be around 200 are (8) and (29).

Organization: In Section 2 we introduce some standard definitions and techniques, and state some useful results from [7]. In Section 3 we make some quick reductions and setup parameters for the families $\mathcal{A}$ and $\mathcal{B}$ by which we break the proof down into cases.

In particular, we introduce a pair of parameters (s, s'), with $0 \leq s' \leq s$, which effectively measures the difference between our families $\mathcal{A}$ and $\mathcal{B}$ and the optimal families $\mathcal{F}_0^t, \mathcal{F}_1^t$ or $\mathcal{F}_2^t$. When (s, s') is one of $(0, 0)$, $(1, 1)$, $(2, 2)$, $(1, 0)$, or $(2, 1)$, then $\mathcal{A}$ and $\mathcal{B}$ will be, or will be very close to, one of these families. In this case we have to look closely at the structure of our families, and compare them with the optimal families directly. This will be done in Section 5.

The remaining cases are dealt with in Section 4. When (s, s') is not one of the above five values, then $\mathcal{A}$ and $\mathcal{B}$ are very different from the optimal families, so we can expect them to have relatively small weight. This seems as though it should

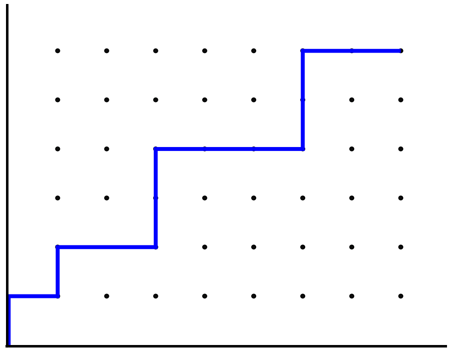

Fig. 1. The walk $F = \{1, 3, 6, 7, 11, 12\} \subset [14]$.

make computation easier, but there is an added difficulty in that we can no longer compute their weight relative to the optimal families, rather we must compute these weights directly. That said, if s is big, a fairly crude estimation of the weight will suffice, and these cases are done in Section 4.1. For the intermediate values of s we consider a finer bound on the size of the families, and use its monotonicity on the range $2 \le s' \le s \le 10$ to achieve our bound for most of these values. This finer bound is still too crude for the final five cases.

This overall approach is based on the paper [7], but the monotonicity ideas used in Section 4.2 are new. We feel that such ideas will be necessary in proving Conjecture 1 for larger values of r. See [8] for some recent developments on cross-intersecting families in different directions.

2. Preliminaries

2.1. Subset vs. walk on a two-dimensional grid

It is useful to regard a set $F \subset [n]$ as a walk starting at the origin $(0, 0)$ of the two-dimensional grid $\mathbb{Z}^2$ as follows. If $i \in F$, then the ith step is *up* from (x, y) to $(x, y + 1)$. Otherwise, the ith step is *right* from (x, y) to $(x + 1, y)$. For simplicity, we refer to $F \subset [n]$ as a set or a walk. See Fig. 1 for an example.

Let $\mathcal{F}^\ell$ be the family of all walks that hit the line $y = x + \ell$; that is, let

$$\mathcal{F}^\ell = \left\{F \subset [n] :\ |F \cap [j]| \ge \frac{j+\ell}{2} \text{ for some } j\right\}.$$

Partition the family $\mathcal{F}^\ell$ into the following three subfamilies:

$$\tilde{\mathcal{F}}^\ell := \left\{F \in \mathcal{F}^\ell :\ F \text{ hits } y = x + \ell + 1\right\},$$
$$\dot{\mathcal{F}}^\ell := \left\{F \in \mathcal{F}^\ell :\ F \text{ hits } y = x + \ell \text{ exactly once, but does not hit } y = x + \ell + 1\right\},$$
$$\ddot{\mathcal{F}}^\ell := \left\{F \in \mathcal{F}^\ell :\ F \text{ hits } y = x + \ell \text{ at least twice, but does not hit } y = x + \ell + 1\right\}.$$

So we can write

$$\mathcal{F}^\ell = \tilde{\mathcal{F}}^\ell \sqcup \dot{\mathcal{F}}^\ell \sqcup \ddot{\mathcal{F}}^\ell.$$

The following lemmas hold.

Lemma 2 *([7], Lemma 2.2 (i, iii)). For any positive integer ℓ, we have the following, where $\alpha = p/q$.*

(i) $\mu_p(\mathcal{F}^\ell) \le \alpha^\ell$ *and* $\mu_p(\tilde{\mathcal{F}}^\ell) \le \alpha^{\ell+1}$
(ii) $\mu_p(\ddot{\mathcal{F}}^\ell) \le \alpha^{\ell+1}$.

Lemma 3 *([7], Lemma 2.2 (ii)). For every $\epsilon > 0$, there exists an n_0 such that if n and l are integers satisfying $n \ge n_0$ and $l \ge 1$, then the following holds: If $\mathcal{F} \subset 2^{[n]}$ and no walk in $\mathcal{F}$ hits the line $y = x + \ell$, then*

$$\mu_p(\mathcal{F}) < 1 - \alpha^\ell + \epsilon.$$

2.2. Inclusion maximal and shifted families

A family $\mathcal{F} \subset 2^{[n]}$ is called *inclusion maximal* if $F \in \mathcal{F}$ and $F \subset F'$ imply $F' \in \mathcal{F}$.

Fact 4. *If $\mathcal{A}$, $\mathcal{B}$ are cross t-intersecting families in $2^{[n]}$, then there are inclusion maximal cross t-intersecting families $\mathcal{A}'$, $\mathcal{B}' \in 2^{[n]}$ such that $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{B} \subset \mathcal{B}'$.*

For $F \subset [n]$ and $i, j \in [n]$, let

$$s_{ij}(F) := \begin{cases} (F \setminus \{j\}) \cup \{i\} & \text{if } F \cap \{i, j\} = \{j\} \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

Then, for $\mathcal{F} \subset 2^{[n]}$, let

$$s_{ij}(\mathcal{F}) := \{s_{ij}(F) : F \in \mathcal{F}\}.$$

A family $\mathcal{F}$ is called *shifted* if $s_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \le i < j \le n$. Here we list some basic properties concerning shifting operations.

Lemma 5 *([7], Lemma 2.3). Let $1 \le i < j \le n$ and let $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$.*

(i) *Shifting operations preserve the p-weight of a family, that is, $\mu_p(s_{ij}(\mathcal{G})) = \mu_p(\mathcal{G})$.*
(ii) *If $\mathcal{F}$ and $\mathcal{G}$ in $2^{[n]}$ are cross t-intersecting families, then $s_{ij}(\mathcal{F})$ and $s_{ij}(\mathcal{G})$ are cross t-intersecting families as well.*
(iii) *For a pair of families we can always obtain a pair of shifted families by repeatedly shifting families simultaneously finitely many times.*

The following lemma, which mimics a proposition in [1], that is in turn based on an idea from [5], was stated in [7], but its proof was only sketched. We prove it now for all values of r, though we only need it for $r = 0$, 1, and 2.

Lemma 6. *Let $t \ge 2$ and let $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ be cross t-intersecting families. If $s_{ij}(\mathcal{A}) = s_{ij}(\mathcal{B}) = \mathcal{F}_r^t$, then $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_r^t$.*

Proof. First we remark that if $\mathcal{A} \cong \mathcal{F}_r^t$, then $\mathcal{B} = \mathcal{A}$. Further, if $i, j \in [t + 2r]$ or $i, j \notin [t + 2r]$, then $\mathcal{A} = \mathcal{F}_r^t$ and we are done. So without loss of generality we may assume that $i = t + 2r$ and $j = n$. Define two subfamilies $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ by

$$\mathcal{A}_1 = \{A \in \mathcal{A} : |A| = t + r,\ i \notin A,\ j \in A,\ (A \cup \{i\}) \setminus \{j\} \notin \mathcal{A}\},$$
$$\mathcal{A}_2 = \{A \in \mathcal{A} : |A| = t + r,\ i \in A,\ j \notin A,\ (A \setminus \{i\}) \cup \{j\} \notin \mathcal{A}\}.$$

Since $s_{ij}(\mathcal{A}) = \mathcal{F}_r^t$, we have $|A \cap [t + 2r - 1]| = t + r - 1$ for all $A \in \mathcal{A}_1 \cup \mathcal{A}_2$. If $\mathcal{A}_1 = \emptyset$, then $\mathcal{A} = \mathcal{F}_r^t$. If $\mathcal{A}_2 = \emptyset$, then $\mathcal{A} = \{A \subset [n] : |A \cap ([i - 1] \cup \{j\})| \ge t + r\} \cong \mathcal{F}_r^t$. So we may assume that $\mathcal{A}_1 \ne \emptyset$ and $\mathcal{A}_2 \ne \emptyset$.

Let $\mathcal{H} = \binom{[t+2r-1]}{t+r-1}$. Then for every $H \in \mathcal{H}$ we have $H \cup \{j\} \in \mathcal{A}_1$ or $H \cup \{i\} \in \mathcal{A}_2$ (but not both). Thus we can identify $\mathcal{H}$ with $\mathcal{A}_1 \sqcup \mathcal{A}_2$. We also define $\mathcal{B}_1$ and $\mathcal{B}_2$ in the same manner. Let $\mathcal{H}'$ be a copy of $\mathcal{H}$, and identify $\mathcal{H}'$ with $\mathcal{B}_1 \sqcup \mathcal{B}_2$.

Now we define a bipartite graph G on $V(G) = \mathcal{H} \sqcup \mathcal{H}'$, by letting $\{H, H'\}$ be an edge, for $H \in \mathcal{H}$ and $H' \in \mathcal{H}'$, if $|H \cap H'| = t - 1$. We claim that G is a connected graph. Indeed, the graph G_0 defined on $\mathcal{H}$ by letting $\{H_1, H_2\}$ be an edge if $|H_1 \cap H_2| = t - 1$, is Kneser's graph; and this is connected and non-bipartite for $t > 1$. If G is not connected, then each connected component is isomorphic to G_0, which contradicts the fact that G_0 is not bipartite. This shows that G is connected.

Therefore, there is a path from $A \in \mathcal{A}_1$ to $B \in \mathcal{B}_2$ in G, and on this path there is an edge $\{A_1, B_2\}$ where $A_1 \in \mathcal{A}_1, B_2 \in \mathcal{B}_2$, or an edge $\{A_2, B_1\}$ where $A_2 \in \mathcal{A}_2, B_1 \in \mathcal{B}_1$. But then $|A_1 \cap B_2| = t - 1$ or $|A_2 \cap B_1| = t - 1$, which contradicts the fact that $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting. $\square$

Fact 4 and Lemma 5 allow us to assume that $\mathcal{A}$ and $\mathcal{B}$ are inclusion maximal and shifted in proving the inequalities in Theorem 1. Lemma 6, allows us to extend this assumption to the uniqueness results in the case of equality in the theorem. We record this as the following assumption.

Assumption 7. $\mathcal{A}$ and $\mathcal{B}$ are inclusion maximal and shifted.

3. Setup for proof of Theorem 1

Recall that n and t are integers with $n \ge t \ge 200$, and p is a real number with $\frac{1}{t+1} \le p \le \frac{2}{t+3}$. Set $q = 1 - p$ and $\alpha = p/q$. The following holds.

Lemma 8 *([7], Lemma 2.12). Let $f(n)$ be the maximum of $\mu_p(\mathcal{A})\mu_p(\mathcal{B})$ over all pairs $\mathcal{A}$ and $\mathcal{B}$ of cross t-intersecting families in $2^{[n]}$. Then, $f(n) \le f(n + 1)$.*

We may therefore assume that n is arbitrarily large.

For $\mathcal{F} \subset 2^{[n]}$, let $\lambda(\mathcal{F})$ be the maximum integer $\lambda \ge 0$ such that all walks in $\mathcal{F}$ hit the line $y = x + \lambda$. Let $u = \lambda(\mathcal{A})$ and $v = \lambda(\mathcal{B})$. The following holds.

Lemma 9 *([7], Lemma 2.11(ii)). If $\mathcal{A}$ and $\mathcal{B}$ are shifted, inclusion maximal, cross t-intersecting families in $2^{[n]}$, then $\lambda(A) + \lambda(B) \ge 2t$.*

Therefore, we assume that $u+v \geq 2t$. If $u+v \geq 2t+1$, then Lemma 2 gives that

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq \mu_p(\mathcal{F}^u)\mu_p(\mathcal{F}^v) \leq \alpha^u\alpha^v \leq \alpha^{2t+1}.$$

One can check that $\alpha^{2t+1} < 0.99\big(\mu_p(\mathcal{F}_1^t)\big)^2$ for $t \geq 26$. Indeed,

$$\frac{\big(\mu_p(\mathcal{F}_1^t)\big)^2}{\alpha^{2t+1}} \geq (t+2)^2pq^3(q^t)^2 \geq (t+2)^2pq^3\frac{1}{e^4},$$

where the second inequality follows from

$$\frac{1}{e^2} < \left(1-\frac{2}{t+3}\right)^{t+3} < \left(\frac{t+1}{t+3}\right)^t \leq q^t \leq \left(\frac{t}{t+1}\right)^t \leq 0.5. \tag{4}$$

Since pq^3 is increasing in p for $p \leq 0.25$, we have that

$$\frac{\big(\mu_p(\mathcal{F}_1^t)\big)^2}{\alpha^{2t+1}} \geq \frac{(t+2)^2t^3}{e^4(t+1)^4} > 1.02,$$

where the last inequality holds for $t \geq 26$.

Therefore, we assume that

$$u+v = 2t.$$

Without loss of generality, let

$$u \leq v.$$

Note that $\mathcal{A} \subset \mathcal{F}^u$. So $\mathcal{A}$ is partitioned as $\mathcal{A} = \tilde{\mathcal{A}} \sqcup \dot{\mathcal{A}} \sqcup \ddot{\mathcal{A}}$, where

$$\tilde{\mathcal{A}} := \mathcal{A} \cap \tilde{\mathcal{F}}^u, \qquad \dot{\mathcal{A}} := \mathcal{A} \cap \dot{\mathcal{F}}^u, \quad \text{and} \quad \ddot{\mathcal{A}} := \mathcal{A} \cap \ddot{\mathcal{F}}^u.$$

Similarly, we have that $\mathcal{B} = \tilde{\mathcal{B}} \sqcup \dot{\mathcal{B}} \sqcup \ddot{\mathcal{B}}$, where

$$\tilde{\mathcal{B}} := \mathcal{B} \cap \tilde{\mathcal{F}}^v, \qquad \dot{\mathcal{B}} := \mathcal{B} \cap \dot{\mathcal{F}}^v, \quad \text{and} \quad \ddot{\mathcal{B}} := \mathcal{B} \cap \ddot{\mathcal{F}}^v.$$

If $\dot{\mathcal{A}} = \emptyset$, then $\mathcal{A} = \ddot{\mathcal{A}} \cup \tilde{\mathcal{A}}$, and hence,

$$\mu_p(\mathcal{A}) = \mu_p(\tilde{\mathcal{A}}) + \mu_p(\ddot{\mathcal{A}}) \leq \mu_p(\tilde{\mathcal{F}}^u) + \mu_p(\ddot{\mathcal{F}}^u) \leq \alpha^{u+1} + \alpha^{u+1},$$

where the last inequality follows from Lemma 2. Thus, we have that

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq (\alpha^{u+1} + \alpha^{u+1})\alpha^v \leq 2\alpha^{2t+1} < 0.99\left(\mu_p(\mathcal{F}_1^t)\right)^2,$$

where the last inequality holds for $t \geq 110$. Similarly, we have that if $\dot{\mathcal{B}} = \emptyset$, then, for $t \geq 110$,

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) < 0.99\left(\mu_p(\mathcal{F}_1^t)\right)^2.$$

So (3) holds if $\dot{\mathcal{A}} = \emptyset$ or $\dot{\mathcal{B}} = \emptyset$.

We may therefore assume that $\dot{\mathcal{A}} \neq \emptyset$ and $\dot{\mathcal{B}} \neq \emptyset$. Recall that

$$\mathcal{F}_i^\ell = \Big\{F \subset [n] : |F \cap [\ell+2i]| \geq \ell + i\Big\}.$$

That is, $\mathcal{F}_i^\ell$ is the family of walks hitting $(i, i+k)$ for some $k \geq \ell$. Note that as $\dot{\mathcal{A}}$ and $\dot{\mathcal{B}}$ are non-empty, there exist non-negative integers s and s' such that $\dot{\mathcal{A}} \cap \mathcal{F}_s^u \neq \emptyset$ and $\dot{\mathcal{B}} \cap \mathcal{F}_{s'}^v \neq \emptyset$. The next lemma tells us that such s and s' are unique. Its statement has been modified, but it is essentially Lemma 3.2 of [7].

Lemma 10 *([7], Lemma 3.2)*. *Suppose that $\dot{\mathcal{A}} \neq \emptyset$ and $\dot{\mathcal{B}} \neq \emptyset$. Then, there exist unique non-negative integers s and s' such that*

$$\mathcal{A}_s := \dot{\mathcal{A}} \sqcup \ddot{\mathcal{A}} \subset \mathcal{F}_s^u \quad \text{and} \quad \mathcal{B}_{s'} := \dot{\mathcal{B}} \sqcup \ddot{\mathcal{B}} \subset \mathcal{F}_{s'}^v.$$

Moreover, $s - s' = (v-u)/2$. In particular, $s \geq s'$.

Here, we record our **setup**.

- $\mathcal{A}$ and $\mathcal{B}$ are shifted maximal cross t-intersecting families.
- n may be assumed to be arbitrarily large.
- $q = 1-p$ and $\alpha = p/q$.
- $t \geq 200$, and $\frac{1}{t+1} \leq p \leq \frac{2}{t+3}$, so $\frac{t+1}{t+3} \leq q \leq \frac{t}{t+1}$.
- $u+v = 2t$ and $1 \leq u \leq t \leq v \leq 2t$.
- $s \geq s' \geq 0$ and $s - s' = (v-u)/2$.
- $u = t - s + s'$ and $v = t + s - s'$.
- $\mathcal{A} = \dot{\mathcal{A}} \sqcup \ddot{\mathcal{A}} \sqcup \tilde{\mathcal{A}} \subset \mathcal{F}^u$ and $\mathcal{B} = \dot{\mathcal{B}} \sqcup \ddot{\mathcal{B}} \sqcup \tilde{\mathcal{B}} \subset \mathcal{F}^v$.
- $\dot{\mathcal{A}} \neq \emptyset$, $\dot{\mathcal{B}} \neq \emptyset$, $\dot{\mathcal{A}} \sqcup \ddot{\mathcal{A}} \subset \mathcal{F}_s^u$, and $\dot{\mathcal{B}} \sqcup \ddot{\mathcal{B}} \subset \mathcal{F}_{s'}^v$.

4. Almost all cases

The rest of the proof is broken down into cases based on the value of (s, s'). We first deal with the cases with $s \geq 10$. Then we spend the rest of the section reducing the remaining cases to the five final cases which will be proved in Section 5.

4.1. Large values of s

Let

$$\bar{\mathcal{F}}_i^{\ell} := (\dot{\mathcal{F}}^{\ell} \cup \ddot{\mathcal{F}}^{\ell}) \cap \mathcal{F}_i^{\ell}. \tag{5}$$

We use the following key estimation from [7].

Claim 11 ([7], Claim 3.3)**.** *There is an integer n_0 such that if $n > n_0$, then*

$$\mu_p(\tilde{\mathcal{F}}^{\ell} \cup \bar{\mathcal{F}}_i^{\ell}) < f(\ell, i, p) \cdot 1.001,$$

where

$$f(\ell, i, p) := \alpha^{\ell+1} + \binom{\ell+2i}{i}\frac{\ell+1}{\ell+i+1}p^{\ell+i}q^i(1-\alpha). \tag{6}$$

Now, since $\mathcal{A} \subset \tilde{\mathcal{F}}^u \cup \bar{\mathcal{F}}_s^u$ and $\mathcal{B} \subset \tilde{\mathcal{F}}^v \cup \bar{\mathcal{F}}_{s'}^v$, we have that

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq \mu_p(\tilde{\mathcal{F}}^u \cup \bar{\mathcal{F}}_s^u)\mu_p(\tilde{\mathcal{F}}^v \cup \bar{\mathcal{F}}_{s'}^v) \leq f(u, s, p)f(v, s', p) \cdot 1.001^2.$$

Hence, in order to show (3), it suffices to show that

$$g(s, s') := f(u, s, p)f(v, s', p) < 0.99\left(\mu_p(\mathcal{F}_1^t)\right)^2.$$

Observe that $g(s, s')$ depends on t, p, s and s', but for simplicity we only write the variables s and s'.

Claim 12. *For $s \geq 10$, we have $g(s, s') < 0.99\left(\mu_p(\mathcal{F}_1^t)\right)^2$.*

Proof. Set $h(\ell, i, p) := p^{-\ell}f(\ell, i, p)$, so that

$$h(\ell, i, p) = \frac{p}{q^{\ell+1}} + \binom{\ell+2i}{i}\frac{\ell+1}{\ell+i+1}(pq)^i(1-\alpha).$$

Since $p^{2t} \leq \left(\mu_p(\mathcal{F}_1^t)\right)^2$, it suffices to show that

$$h(u, s, p)h(v, s', p) < 0.99. \tag{7}$$

First, we estimate $h(u, s, p)$. We have that

$$h(u, s, p) \leq h(t, s, p) \leq h\left(t, s, \frac{2}{t+3}\right),$$

where the second inequality holds since p, $1/q$, pq, and $pq(1-\alpha)$ are increasing in p for $0 < p \leq 2/(t+3) \leq 0.25$. Consequently, since $p/q^{t+1} = \frac{2}{t+3}(\frac{t+3}{t+1})^{t+1} \leq 0.073$ for $t \geq 200$, we have

$$\begin{aligned} h\left(t, s, \frac{2}{t+3}\right) &\leq 0.073 + \binom{t+2s}{s}\left(\frac{2}{t+3}\right)^s \leq 0.073 + \left(\frac{e(t+2s)}{s}\frac{2}{t+3}\right)^s \\ &= 0.073 + \left(\frac{2e(t+2s)}{s(t+3)}\right)^s. \end{aligned}$$

Note that $\frac{2e(t+2s)}{s(t+3)} < 1$ if and only if $s > \frac{2et}{t+3-4e}$. Since $s \geq 6 > \frac{2et}{t+3-4e}$ for $t \geq 84$, we have that $\frac{2e(t+2s)}{s(t+3)} < 1$. Also, $\phi(s) := \frac{2e(t+2s)}{s(t+3)}$ is strictly decreasing in s, since $\frac{\phi(s+1)}{\phi(s)} = \frac{(t+2s+2)s}{(t+2s)(s+1)} < 1$. Therefore, for $s \geq 10$,

$$h\left(t, s, \frac{2}{t+3}\right) \leq 0.073 + \left(\frac{2e(t+20)}{10(t+3)}\right)^{10} < 0.08. \tag{8}$$

Next, we estimate $h(v, s', p)$. Similar to the estimation of $h(u, s, p)$, we have that

$$h(v, s', p) \leq h(2t, s', p) \leq h\left(2t, s', \frac{2}{t+3}\right).$$

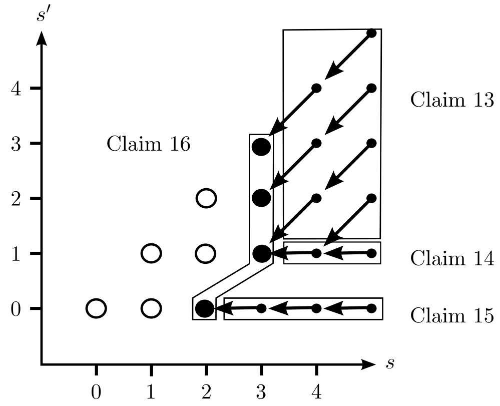

Fig. 2. $(s_1, s_1') \to (s_2, s_2')$ means $g(s_1, s_1') < g(s_2, s_2')$.

Consequently, since $p/q^{2t+1} \leq 0.53$ for $t \geq 200$, we infer that

$$h\left(2t, s', \frac{2}{t+3}\right) \leq 0.53 + \binom{2t+2s'}{s'}\left(\frac{2}{t+3}\right)^{s'} \leq 0.53 + \psi_a\psi_b,$$

where $\psi_a = 4^{s'}/s'!$ and $\psi_b = \frac{(t+s)}{t+3}\frac{t+s-1/2}{t+3}\cdots\frac{t+(s+1)/2}{t+3}$. Now $\psi_a = 64/6 < 10.7$ for $s' = 3, 4$ and is otherwise less than 8.54. On the other hand, ψ_b is less than 1 for $s' \leq 3$, and is decreasing in t for $s' \geq 4$. Thus for $s' \leq 3$, $\psi_a\psi_b < 10.7$. Using its value at $t = 100$ to bound ψ_b for $s' = 4, \ldots, 25$ we get that $\psi_a\psi_b < 10.77$. As $\psi_b < \left(\frac{t+s'}{t+3}\right)^{s'} < e^{\frac{s'(s'-1)}{t+3}} < e^{s'}$, we have for $s' > 25$ that $\psi_a\psi_b < (4e)^{s'}/s'! < 4e^{25}/25! < 6$. So for $s' \geq 0$, we get

$$h\left(2t, s', \frac{2}{t+3}\right) < .53 + 10.77 = 11.3. \tag{9}$$

Therefore, we have that

$$h(u, s, p)h(v, s', p) < h\left(t, s, \frac{2}{t+3}\right)h\left(2t, s', \frac{2}{t+3}\right) \overset{(8),(9)}{<} 0.08 \cdot 11.3 < 0.99,$$

which yields (7). $\square$

4.2. Intermediate values of s

The remaining cases of (s, s') are $0 \leq s' \leq s \leq 9$. In this section we deal with all but five of these. We do this by showing the monotonicity of $g(s, s')$ on several ranges, and then bounding $g(s, s')$ for four particular cases. See Fig. 2 for a schematic of the proof. In Claim 13 we show $g(s, s') < g(s-1, s'-1)$ for values of (s, s') as indicated in the figure, (actually for more values, but we only use those indicated in the figure). In Claims 14 and 15 we show that $g(s, s') < g(s-1, s')$ for the values $s = 0$ and 1 as indicated. In Claim 16 we show that $g(s, s') < 0.99\left(\mu_p(\mathcal{F}_1^t)\right)^2$ in the cases that $(s, s') = (3, 3), (3, 2), (3, 1)$ and $(2, 0)$.

The final five values of (s, s'), the empty dots, are dealt with in Section 5.

Claim 13. *For $t \geq 10$ and $2 \leq s' \leq s \leq 9$ with $(s, s') \neq (2, 2)$ we have*

$$g(s, s') < g(s-1, s'-1).$$

Proof. Recall that $g(s, s') = f(u, s, p)f(v, s', p)$ and $g(s-1, s'-1) = f(u, s-1, p)f(v, s'-1, p)$. Hence, it suffices to show that

$$f(u, s, p) < f(u, s-1, p) \tag{10}$$

and

$$f(v, s', p) < f(v, s'-1, p). \tag{11}$$

First, inequality (10) is equivalent to

$$\binom{u+2s}{s}\frac{pq}{u+s+1} < \binom{u+2s-2}{s-1}\frac{1}{u+s}.$$

We have that

$$\left(\binom{u+2s}{s}\frac{pq}{u+s+1}\right)\Big/\left(\binom{u+2s-2}{s-1}\frac{1}{u+s}\right) \le \frac{(u+2s)(u+2s-1)\cdot 2(t+1)}{s(u+s+1)\cdot(t+3)^2}$$
$$= \frac{2(t+s+s')(t+s+s'-1)(t+1)}{s(t+s'+1)(t+3)^2}$$
$$< \frac{2(t+s+s')(t+s+s'-1)}{s(t+s'+1)(t+3)}.$$

This is decreasing in t as $s+s' \ge 4$, so setting $t = 10$ and computing casewise, we get that it is less than .88 for $2 \le s' \le s \le 9$ and $(s', s) \ne (2, 2)$.

Similarly, inequality (11) is equivalent to

$$\binom{v+2s'}{s'}\frac{pq}{v+s'+1} < \binom{v+2s'-2}{s'-1}\frac{1}{v+s'}.$$

We have that

$$\left(\binom{v+2s'}{s'}\frac{pq}{v+s'+1}\right)\Big/\left(\binom{v+2s'-2}{s'-1}\frac{1}{v+s'}\right) \le \frac{2(v+2s')(v+2s'-1)(t+1)}{s'(v+s'+1)(t+3)^2}$$
$$< \frac{2(t+s+s')(t+s+s'-1)}{s'(t+s+1)(t+3)}.$$

This is again decreasing in t, and with $t = 10$ we compute that it is less than .88 for $2 \le s' \le s \le 9$ and $(s', s) \ne (2, 2)$. (The maximum value is at $(s, s') = (3, 3)$, which is why it is the same value as above.) $\square$

Claim 14. *For $s \ge 2$ and $s' = 1$, we have $g(s, 1) < g(1, 1)$.*

Proof. Note that $u = t - s + 1$ and $v = t + s - 1$. Recalling (6), we can write

$$p^s f(u, s, p) = C_1 q^s + C_2 h(s) q^s, \quad \text{and}$$
$$p^{-s} f(v, 1, p) = C_3 q^{-s} + C_4(t+s),$$

where $h(s) := \binom{t+s+1}{s}(t-s+2)p^s$ and

$$C_1 = \alpha^t, \qquad C_2 = \frac{p^{t-1}(1-\alpha)}{t+3}, \qquad C_3 = \alpha^t, \quad \text{and} \quad C_4 = p^t q(1-\alpha).$$

Note that $C_1, C_2, C_3, C_4 > 0$ depend only on t and p (and do not depend on s).

Multiplying $p^s f(u, s, p)$ and $p^{-s} f(v, 1, p)$, we have that

$$g(s, 1) = D_1 + D_2 q^s(t+s) + D_3 h(s) + D_4 h(s) q^s(t+s), \tag{12}$$

where $D_1, D_2, D_3, D_4 > 0$ depend only on t and p.

We claim that $g(s, 1)$ is strictly decreasing in s for $s \ge 1$. By (12), it suffices to show that $q^s(t+s)$ and $h(s)$ are strictly decreasing in s. First, $q^s(t+s)$ is strictly decreasing in s since

$$\frac{q^{s+1}(t+s+1)}{q^s(t+s)} = \frac{q(t+s+1)}{(t+s)} \le \frac{t(t+s+1)}{(t+1)(t+s)} < 1,$$

where the first inequality follows from $q \le t/(t+1)$ and the last inequality holds for $s \ge 1$. Next, $h(s)$ is strictly decreasing in s since

$$\frac{h(s+1)}{h(s)} \le \frac{2(t+s+2)(t-s+1)}{(s+1)(t-s+2)(t+3)} < \frac{2(t+s+2)}{(s+1)(t+3)} \le 1,$$

where the first inequality follows from $p \le 2/(t+3)$ and the last inequality follows from $s \ge 1$. $\square$

Claim 15. *For $s \ge 2$ and $s' = 0$, we have $g(s, 0) < g(1, 0)$.*

Proof. Again, noting this time that $u = t - s$ and $v = t + s$, we write

$$p^s f(u, s, p) = C_1 q^s + C_2 h(s) q^s, \quad \text{and}$$
$$p^{-s} f(v, 0, p) = C_3 q^{-s} + C_4,$$

where $h(s) := \binom{t+s}{s}(t-s+1)p^s$, and $C_1, C_2, C_3, C_4 > 0$ (different from above) depend only on t and p. Multiplying $p^s f(u, s, p)$ and $p^{-s} f(v, 0, p)$, we have that

$$g(s, 0) = D_1 + D_2 q^s + D_3 h(s) + D_4 h(s) q^s, \tag{13}$$

where $D_1, D_2, D_3, D_4 > 0$ depend only on t and p.

We claim that $g(s, 0)$ is strictly decreasing in s for $s \geq 1$. By (13), it suffices to show that $h(s)$ is strictly decreasing in s. Indeed,

$$\frac{h(s+1)}{h(s)} \leq \frac{2(t+s+1)(t-s)}{(s+1)(t-s+1)(t+3)} < \frac{2(t+s+1)}{(s+1)(t+3)} \leq 1,$$

where the first inequality follows from $p \leq 2/(t+3)$ and the last inequality holds for $s \geq 1$. $\square$

Claim 16. *For $t \geq 52$ and $(s, s') = (3, 3), (3, 2), (3, 1)$ and $(2, 0)$ we have $g(s, s') < 0.99 \left(\mu_p(\mathcal{F}_1^t)\right)^2$.*

Proof. We give the calculations for the case $(s, s') = (3, 1)$. The calculations for the other cases are very similar, and given in the Appendix. For the estimation in all cases we use $e^{-2} \leq q^t$, and $q^{-i} = (\frac{t+3}{t+1})^i < 2$ for $1 \leq i \leq 6$ and $t \geq 16$. Noting that $u = t - 2$ and $v = t + 2$ we get that

$$f(u, 3, p) = \alpha^{t-1} + \binom{t+4}{3}\frac{t-1}{t+2}p^{t+1}q^3(1-\alpha) < e^2\frac{p^{t-1}}{q} + \frac{(t+4)(t+3)(t-1)}{6}p^{t+1}q^3,$$

and we get

$$f(v, 1, p) = \alpha^{t+3} + (t+4)\frac{t+3}{t+4}p^{t+3}q(1-\alpha) < \frac{e^2 p^{t+3}}{q^3} + (t+3)p^{t+3}q.$$

Thus as $(\mu_p(\mathcal{F}_1^t))^2 > (t+2)^2 p^{2t+2} q^2$ we get that

$$\begin{aligned}\frac{g(3,1)}{(\mu_p(\mathcal{F}_1^t))^2} &< \frac{e^4}{q^6(t+2)^2} + \frac{e^2(t+3)}{(t+2)^2q^2} + \frac{e^2p^2(t+4)(t+3)(t-1)}{6q^2(t+2)^2} + \frac{p^2q^4(t+4)(t+3)^2}{6(t+2)^2} \\ &< \frac{2e^4}{(t+2)^2} + \frac{2e^2(t+3)}{(t+2)^2} + \frac{8e^2(t+4)(t+3)(t-1)}{6(t+2)^2(t+3)^2} + \frac{4(t+4)(t+3)^2}{6(t+2)^2(t+3)^2}.\end{aligned}$$

This is less than .99 for $t \geq 28$. $\square$

Referring to Fig. 2, or our outline of the proof preceding Claim 13, Claims 12–15 imply the following corollary.

Corollary 17. *If $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq \left(\mu_p(\mathcal{F}_1^t)\right)^2$ holds for*

$$(s, s') = (0, 0), (1, 0), (1, 1), (2, 2),$$

then, for all (s, s') other than $(0, 0), (1, 0), (1, 1), (2, 2)$,

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) < \left(\mu_p(\mathcal{F}_1^t)\right)^2.$$

5. Remaining cases

5.1. Definitions

We introduce several definitions and notation. For $A \subset [n]$, let $(A)_i$ be the ith smallest element of A. For $A, B \subset [n]$, we say that A *shifts to* B, denoted by

$$A \to B,$$

if $|A| \leq |B|$ and $(A)_i \geq (B)_i$ for each $i \leq |A|$. In other words, as walks on a two-dimensional grid, each edge of the walk B is not contained in the area to the right of the walk A. For example, $\{2, 4, 6\} \to \{1, 4, 5, 7\}$.

Fact 18 *([7], Fact 2.8). Let $\mathcal{F}$ be a shifted, inclusion maximal family in $2^{[n]}$. If $F \in \mathcal{F}$ and $F \to F'$, then $F' \in \mathcal{F}$.*

This immediately implies the following.

Fact 19. *Let $\mathcal{F}$ be a shifted, inclusion maximal family in $2^{[n]}$. If $F' \notin \mathcal{F}$, then every $F \in \mathcal{F}$ satisfies $F \not\to F'$.*

For $t \in [n]$ and $F \subset [n]$, the *dual* of F with respect to t is defined by

$$\mathrm{dual}_t(F) := [(F)_t - 1] \cup ([n] \setminus F).$$

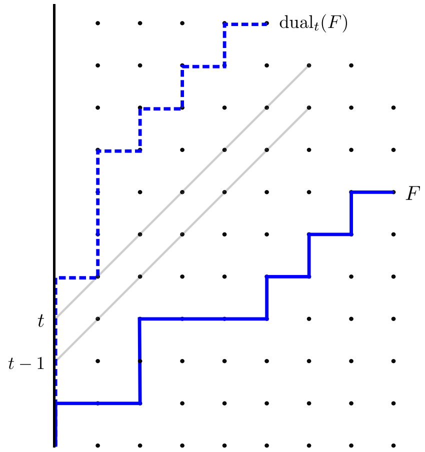

Fig. 3. The walk $\mathrm{dual}_t(F)$ of F with respect to t.

Viewed as walks on a two-dimensional grid, the walk $\mathrm{dual}_t(F)$ is obtained by reflecting F across the line $y = x + (t-1)$ and ignoring the part $x < 0$. (See Fig. 3.)

The dual $\mathrm{dual}_t(F)$ of a set F is defined so that its intersection with $[t, n]$ is the complement of that of F, so $|F \cap \mathrm{dual}_t(F)| < t$. This gives the following.

Fact 20 *([7], Fact 2.9). Let $\mathcal{A}$ and $\mathcal{B}$ be cross t-intersecting families. If $A \in \mathcal{A}$, then $\mathrm{dual}_t(A) \notin \mathcal{B}$.*

For integers $\ell, i \geq 1$ and $s \geq 0$, let

$$D_s^\ell(i) := [\ell-1] \cup \{\ell-1+2, \ell-1+4, \ldots, \ell-1+2s\} \cup \{\ell+2s\} \cup \{\ell+2s+i+2k \in [n] : k = 1, 2, \ldots\}. \quad (14)$$

This walk is the maximally shifted walk in $\dot{\mathcal{F}}^\ell \cap \mathcal{F}_s^\ell$ with the property that it goes left for $i+1$ steps after hitting the line $y = x + \ell$ at $(s, s+l)$, and then after that does not go above the line $y = x + \ell - i$. Note that $D_s^\ell(i) = D_s^\ell(n-\ell-2s-1)$ for $i \geq n - \ell - 2s - 1$, and hence, we assume that

$$1 \leq i \leq n - \ell - 2s - 1. \quad (15)$$

The walks $D_1^{t-1}(i)$ and $D_0^{t+1}(j)$ are denoted by $D_i^{\mathcal{A}}$ and $D_j^{\mathcal{B}}$ in [7]. (They are depicted in Figure 1 of [7].)

5.2. *The cases $(s, s') = (2, 1)$ and $(1, 0)$*

Note that in the cases $(s, s') = (2, 1)$ and $(1, 0)$ we have that $u = t - 1$ and $v = t + 1$.

Lemma 21. *Let $t \geq 42$. For $(s, s') = (2, 1)$, we have $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) < \left(\mu_p(\mathcal{F}_1^t)\right)^2$.*

Proof. Consider the following cases of the walks defined in (14). For $1 \leq i \leq n - t - 4$, let

$$D_2^{t-1}(i) = [t-2] \cup \{t, t+2\} \cup \{t+3\} \cup \{t+5+i, t+7+i, t+9+i, \ldots\} \in \dot{\mathcal{F}}^{t-1} \cap \mathcal{F}_2^{t-1},$$

and for $1 \leq j \leq n - t - 4$, let

$$D_1^{t+1}(j) = [t] \cup \{t+2\} \cup \{t+3\} \cup \{t+5+j, t+7+j, t+9+j \ldots\} \in \dot{\mathcal{F}}^{t+1} \cap \mathcal{F}_1^{t+1}.$$

By Fact 18 and the fact that $\dot{\mathcal{A}} \neq \emptyset$ and $\dot{\mathcal{B}} \neq \emptyset$, we have that $D_2^{t-1}(1) \in \mathcal{A}$ and $D_1^{t+1}(1) \in \mathcal{B}$. So the following positive integer values are well defined:

$$I := \max\{i : D_2^{t-1}(i) \in \mathcal{A}\} \quad \text{and} \quad J := \max\{j : D_1^{t+1}(j) \in \mathcal{B}\}.$$

We start with the following general bounds on $\mu_p(\mathcal{A})$ and $\mu_p(\mathcal{B})$, we then show, with casework depending on I and J, that they are sufficient.

Claim 22. *Let $t \geq 20$. For every $\epsilon > 0$ the following holds for sufficiently large n:*

$$\mu_p(\mathcal{A})/p^t < a_1(p, t) + a_2(p, t)\alpha^{J-1} - a_3(p, t)q^{I-2} + \epsilon,$$

where

$$a_1(p,t) := 1 + tpq + \frac{t(t+3)}{2}pq^2,$$
$$a_2(p,t) := q^{-t} - 1 - tpq - \frac{t(t+3)}{2}p^2q < 5,$$
$$a_3(p,t) := \frac{(t+2)(t-1)}{2}pq^5(1-\alpha) > \frac{pq^2}{2}(t^2-7t).$$

Proof. Let $\epsilon > 0$ be given and let $\delta = \epsilon/a_1(p,t)$. As $s = 2$ we have that $\mu_p(\mathcal{A}) = \mu_p(\tilde{\mathcal{A}}) + \mu_p(\mathcal{A}_2)$. To bound $\mu_p(\tilde{\mathcal{A}})$ observe that since $D_1^{t+1}(J) \in \mathcal{B}$, its dual walk

$$\mathrm{dual}_t(D_1^{t+1}(J)) = [t-1] \cup \{t+1\} \cup [t+4, t+J+4] \cup \{t+J+6, t+J+8, \ldots\}$$

is not in $\mathcal{A}$, and all walks in $\mathcal{A}$ must cross it, which is equivalent to hitting $Q_0 := (0,t)$, $Q_1 := (1, t+1)$ or $Q_2 := (2, t+1)$, or hitting the line $L : y = x + (t+J-1)$. Further, walks in $\tilde{\mathcal{A}} \subset \mathcal{F}^u = \mathcal{F}^{t-1}$ all hit the line $L' : y = x + t$. So we have

$$\begin{aligned}\mu_p(\tilde{\mathcal{A}}) \le\ & \mu_p(\text{walks in } \tilde{\mathcal{A}} \text{ hitting } L) + \mu_p(\text{walks in } \tilde{\mathcal{A}} \text{ not hitting } L \text{ but hitting } Q_0)\\ &+ \mu_p(\text{walks in } \tilde{\mathcal{A}} \text{ not hitting } L \text{ or } Q_0 \text{ but hitting } Q_1)\\ &+ \mu_p(\text{walks in } \tilde{\mathcal{A}} \text{ not hitting } L, Q_0 \text{ or } Q_1 \text{ but hitting } Q_2 \text{ and } L').\end{aligned}$$

Using Lemma 2 for the first and last lines, and Lemma 3 for the last three, this gives the following,

$$\mu_p(\tilde{\mathcal{A}}) \le \alpha^{t+J-1} + p^t(1-\alpha^{J-1}+\delta) + tp^{t+1}q(1-\alpha^{J-1}+\delta) + \frac{t(t+3)}{2}p^{t+1}q^2(\alpha - \alpha^J + \delta/2). \tag{16}$$

For the last line we also used that there are $\binom{t+3}{2} - \binom{t+3}{1} = \frac{t(t+3)}{2}$ ways of walks from $(0,0)$ to Q_2 that do not touch the line L'. In fact, Lemma 2.13(ii) of [7] tells us that the number of walks from $(0,0)$ to (x_0, y_0) not hitting the line $y = x + c$ is

$$\binom{x_0+y_0}{x_0} - \binom{x_0+y_0}{y_0-c} \tag{17}$$

for $0 < c < y_0 < x_0 + c$.

Now we bound $\mu_p(\mathcal{A}_2)$. Recall from Lemma 10 and (5) that $\mathcal{A}_2 := \dot{\mathcal{A}} \sqcup \ddot{\mathcal{A}} \subset \bar{\mathcal{F}}_2^{t-1}$. Any walk in $\bar{\mathcal{F}}_2^{t-1} = \mathcal{F}_2^{t-1} \setminus \tilde{\mathcal{F}}^{t-1}$ hits Q_2 without hitting the line L', so without hitting Q_0 or Q_1, and then continues on without hitting L'. So we have

$$\mu_p(\bar{\mathcal{F}}_2^{t-1}) \le \frac{t(t+3)}{2}p^{t+1}q^2(1-\alpha+\delta/2). \tag{18}$$

On the other hand, as $D_2^{t-1}(I+1) \notin \mathcal{A}_2$, $\mathcal{A}_2$ contains no walks in

$$\mathcal{W} = \{W \in \dot{\mathcal{F}}^{t-1} \cap \mathcal{F}_2^{t-1} : W \to D_2^{t-1}(I+1)\}.$$

Such walks hit $(2,t)$ without hitting the line $y = x + (t-1)$, then hit $(2, t+1)$ and then $(I+3, t+1)$ on the line $y = x + (t-I-2)$. After that, they never hit the line $y = x + (t-I-1)$. Using (17) for $x_0 = 2$, $y_0 = t$, $c = t-1$ we have

$$\mu_p(\mathcal{W}) \ge \left(\binom{t+2}{2} - \binom{t+2}{1}\right)p^{t+1}q^{I+3}(1-\alpha) = a_3(p,t)p^tq^{I-2}. \tag{19}$$

We now combine (16), (18) and (19) using the fact that $\mathcal{A} = \tilde{\mathcal{A}} \cup \mathcal{A}_2 \subset \tilde{\mathcal{A}} \cup (\bar{\mathcal{F}}_2^{t-1} \setminus \mathcal{W})$. Observing how nicely (18) combines with the last term in (16), we get

$$\mu_p(\mathcal{A})/p^t \le \frac{\alpha^{J-1}}{q^t} + (1-\alpha^{J-1}+\delta)(1+tpq) + \frac{t(t+3)}{2}pq^2(1-\alpha^J+\delta) - a_3(p,t)q^{I-2}.$$

Rearranging this we get

$$\mu_p(\mathcal{A})/p^t \le (1+\delta)a_1(p,t) + a_2(p,t)\alpha^{J-1} - a_3(p,t)q^{I-1},$$

which is equivalent to the statement of the claim.

To get the bound on $a_2(p,t)$, recall from (4) that $q^{-t} < e^2$ and observe that the other terms in $a_2(p,t)$ are decreasing in p, so for $t \ge 20$ we have

$$a_2(p,t) \le a_2\left(\frac{1}{t+1}, t\right) < e^2 - 1 - \frac{t^2(3t+5)}{2(t+1)^3} < e^2 - 1 - 1.4 < 5.$$

To get the bound on $a_3(p,t)$, observe that $q^3(1-\alpha)$ is decreasing in p, so letting $p=\frac{2}{t+3}$ it follows that

$$q^3(1-\alpha)\geq\left(\frac{t+1}{t+3}\right)^3\left(\frac{t-1}{t+1}\right)=\frac{(t-1)(t+1)^2}{(t+3)^3}$$

and

$$\begin{aligned}a_3(p,t)&=\frac{pq^2}{2}\left((t+2)(t-1)q^3(1-\alpha)\right)\geq\frac{pq^2}{2}(t+2)(t-1)\frac{(t-1)(t+1)^2}{(t+3)^3}\\&=\frac{pq^2}{2}\left(\frac{t^5+2t^4-2t^3-4t^2+t+2}{t^3+9t^2+27t+27}\right)>\frac{pq^2}{2}(t^2-7t),\end{aligned}$$

which gives the bound. $\diamond$

Similarly, we get the following. The proof is in the Appendix.

Claim 23. *For every $\epsilon>0$ the following holds for $t\geq 18$:*

$$\mu_p(\mathcal{B})/p^{t+2}\leq b_1(p,t)+b_2(p,t)\alpha^{I-1}-b_3(p,t)q^{J-2}+\epsilon,$$

where

$$\begin{aligned}b_1(p,t)&:=1+(t+2)q,\\b_2(p,t)&:=q^{-(t+2)}-1-(t+2)p<4.5,\\b_3(p,t)&:=(t+1)q^4(1-\alpha)>(t-7)q.\quad\diamond\end{aligned}$$

To prove the lemma it is now enough to show that

$$(\mu_p(\mathcal{A})/p^t)(\mu_p(\mathcal{B})/p^{t+2})<z^2,$$

where

$$z=\mu_p(\mathcal{F}_1^t)/p^{t+1}=t+2-(t+1)p. \tag{20}$$

We have cases depending on I and J.

- **Case 1.** *Suppose $I\geq 3$ and $J\geq 3$.*

  First observe that for $J\geq 3$ we have

$$\mu_p(\mathcal{A})/p^t<a_1(p,t)+5\alpha^2. \tag{21}$$

Indeed if $J=3$ this is immediate from Claim 22 by taking $\epsilon<(5-a_2(p,t))\alpha^2$. For $J\geq 4$, Claim 22 gives that $\mu_p(\mathcal{A})/p^t<a_1(p,t)+5\alpha^3+\epsilon$. Because $\alpha^3<\alpha^2$, the claim follows by taking $\epsilon<5(\alpha^2-\alpha^3)$. Similarly, it follows from Claim 23 that for $I\geq 3$ and $t\geq 18$ we have

$$\mu_p(\mathcal{B})/p^{t+2}<b_1(p,t)+4.5\alpha^2.$$

So it suffices to show $xy<z^2$ where $x:=a_1(p,t)+5\alpha^2$, $y:=b_1(p,t)+4.5\alpha^2$. One can show that y/z is increasing in p. Clearly x is increasing and z is decreasing. One can also show that y/z is increasing (see Appendix A.3), so it is enough to check the inequality $xy-z^2<0$ at $p=\frac{2}{t+3}$. By direct computation we see that this is true if $t\geq 42$.

- **Case 2.** *Suppose that $I=1$ or 2.*

  By Claim 22 we get

$$\begin{aligned}\mu_p(\mathcal{A})/p^t&<a_1(p,t)+a_2(p,t)\alpha^{J-1}-a_3(p,t)+\epsilon\\&<a_1(p,t)+5-a_3(p,t)\\&<1+tpq+5tpq^2+5<18.\end{aligned}$$

The last inequality uses that pq and pq^2 are increasing in p, so p can be taken as $2/(t+3)$. By Claim 23 we have that $\mu_p(\mathcal{B})/p^{t+2}<b_1(p,t)+4.5=z+q+4.5<z+5$, and so

$$(\mu_p(\mathcal{A})/p^t)(\mu_p(\mathcal{B})/p^{t+2})<18(z+5).$$

Since $18(z+5)<z^2$ if $18\leq z-5$ we see that $z\geq 23$ suffices. Since z is minimized when $p=\frac{2}{t+3}$ and $z\geq t+\frac{4}{t+3}$, it follows that $z\geq 23$ if $t\geq 23$.

• **Case 3.** *Suppose that $J = 1$ or 2.* By Claim 22 we get that

$$\begin{aligned}\mu_p(\mathcal{A})/p^t &< a_1(p,t) + a_2(p,t) < 1 + tpq + \frac{t(t+3)}{2}pq^2 + 5\\ &< 1 + 2 + t + 5 = t + 8.\end{aligned}$$

The third inequality uses that pq and pq^2 are increasing in p so $p = 2/(t+3)$ can be assumed.

From Claim 23 we get that

$$\begin{aligned}\mu_p(\mathcal{B})/p^{t+2} &< b_1(p,t) + b_2(p,t) - b_3(p,t) + \epsilon\\ &< b_1(p,t) + 4.5 - b_3(p,t) < 14.5.\end{aligned}$$

So $(\mu_p(\mathcal{A})/p^t)(\mu_p(\mathcal{B})/p^{t+2}) < 14.5(t+8)$ which is less than z^2 for $t \geq 23$.

This completes the proof for Case 3, and so for the lemma. □

Lemma 24. *Let $t \geq 26$. For $(s, s') = (1, 0)$, we have $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) < \left(\mu_p(\mathcal{F}_1^t)\right)^2$.*

Proof. Again, consider the following particular cases of walks defined in (14). For $1 \leq i \leq n - t - 2$, let

$$D_1^{t-1}(i) = [t-2] \cup \{t\} \cup \{t+1\} \cup \{t+3+i, t+5+i, t+7+i, \ldots\} \in \dot{\mathcal{F}}^{t-1} \cap \mathcal{F}_1^{t-1}.$$

For $1 \leq j \leq n - t - 2$, let

$$D_0^{t+1}(j) = [t+1] \cup \{t+3+j, t+5+j, t+7+j, \ldots\} \in \dot{\mathcal{F}}^{t+1} \cap \mathcal{F}_0^{t+1}.$$

Again the following values are well defined:

$$I := \max\{i : D_1^{t-1}(i) \in \mathcal{A}\} \quad \text{and} \quad J := \max\{j : D_0^{t+1}(j) \in \mathcal{B}\}.$$

Analogous to Claims 22 and 23 we get the following two claims, which are proved in the Appendix.

Claim 25. *For every $\epsilon > 0$ the following holds:*

$$\mu_p(\mathcal{A})/p^t < a_1(p,t) + a_2(p,t)\alpha^{J-1} - a_3(p,t)q^{I-1} + \epsilon,$$

where

$$a_1(p,t) := 1 + tq, \qquad a_2(p,t) := q^{-t} < 7.4, \qquad a_3(p,t) := (t-1)q^3(1-\alpha) > (t-7)q.$$

Claim 26. *For every $\epsilon > 0$ the following holds for $t \geq 20$:*

$$\mu_p(\mathcal{B})/p^{t+2} \leq b_1(p,t) + b_2(p,t)\alpha^{I-1} - b_3(p,t)q^{J-1} + \epsilon,$$

where

$$b_1(p,t) := 1/p, \qquad b_2(p,t) := q^{-(t+2)} < 7.4, \qquad b_3(p,t) := (q^2/p)(1-\alpha) > .75/p.$$

Using these claims, we finish the lemma by considering three cases.

• **Case 1:** *Suppose that $I \geq 2$ and $J \geq 2$.* As (21) followed from Claim 22 for $I, J \geq 3$ we have that for $I, J \geq 2$ and $t \geq 20$, the following inequalities follow from Claims 25 and 26.

$$\begin{aligned}\mu_p(\mathcal{A})/p^t &< 1 + tq + 7.4\alpha =: a,\\ \mu_p(\mathcal{B})/p^{t+2} &< 1/p + 7.4\alpha =: b.\end{aligned}$$

We need to show that $ab/z^2 < 1$, where z is defined in (20). As $z > (t+2)q$ we show that $ab < ((t+2)q)^2$, and it is enough to show that $a < (t+2-0.335)q$ and $b < (t+2+0.335)q$. The former is equivalent to $1 + 7.4\alpha < 1.665q$, which is true for $p < 0.069$. So it holds for $t \geq 26$. The latter is equivalent to $(1/p + 7.4\alpha)/q < t + 2.335$, the left side of which is decreasing in p for $p < .1$. Evaluating it at $p = 1/(t+1)$ we see that it too holds for $t \geq 26$.

• **Case 2:** *Suppose that $I = 1$.* From Claims 25 and 26 we get that

$$\begin{aligned}\mu_p(\mathcal{A})/p^t &< (1 + tq) + a_2(p,t)\alpha^{J-1} - (t-7)q + \epsilon\\ &< 1 + 7q + 7.4 =: a,\\ \mu_p(\mathcal{B})/p^{t+2} &< 1/p + 7.4 =: b.\end{aligned}$$

Again we need to show that $ab/z^2 < 1$. It is enough to show that $c := ab/((t+2)q)^2 < 1$; and indeed, c is decreasing in p so evaluating c at $p = 1/(t+1)$ we see that c is at most $\frac{7(t+1)(5t+42)(11t+6)}{25t^2(t+2)^2} < 1$ for $t \geq 20$.

- **Case 3:** *Suppose that* $J = 1$.
  From Claims 25 and 26 we get that

$$\mu_p(\mathcal{A})/p^t < 1 + tq + 7.4 =: a,$$
$$\mu_p(\mathcal{B})/p^{t+2} < .25/p + 7.4 =: b.$$

Again we show that $ab/q^2 < (t+2)^2$. We have $a/q = 8.4/q + t \le \frac{8.4(t+3)}{t+1} + t$. On the other hand b/q is decreasing in p for $p < 0.15$, and evaluating it at $p = 1/(t+1)$ we have $b/q \le \frac{(t+1)(5t+153)}{20t}$. Using these inequalities we see that $ab/q^2 < (t+2)^2$ for $t \ge 16$. $\square$

5.3. Extremal cases

Finally, we consider the cases $(s, s') = (0, 0), (1, 1), (2, 2)$.

Lemma 27. *For* $(s, s) = (0, 0), (1, 1), (2, 2)$, *we have*

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \le \left(\mu_p(\mathcal{F}_1^t)\right)^2. \tag{22}$$

Moreover, equality holds if and only if one of the following holds:

(i) $\mathcal{A} = \mathcal{B} = \mathcal{F}_0^t$ *and* $p = \frac{1}{t+1}$,
(ii) $\mathcal{A} = \mathcal{B} = \mathcal{F}_1^t$ *and* $\frac{1}{t+1} \le p \le \frac{2}{t+3}$,
(iii) $\mathcal{A} = \mathcal{B} = \mathcal{F}_2^t$ *and* $p = \frac{2}{t+3}$.

Proof. In these cases, we have $u = v = t$. Recalling (14) and (15), we let

$$\begin{aligned} D_s^t(i) :=& [t-1] \cup \{t+s, t+s+1, \ldots, t+2s\} \\ & \cup \{t+2s+i+2k \in [n] : k = 1, 2, \ldots\} \in \dot{\mathcal{F}}^t \cap \mathcal{F}_s^t \end{aligned}$$

for $1 \le i \le n - t - 2s - 1 =: i_{\max}$.
In order to define

$$I = \max\{i : D_s^t(i) \in \mathcal{A}\} \quad \text{and} \quad J = \max\{i : D_s^t(i) \in \mathcal{B}\},$$

the sets $\{i : D_s^t(i) \in \mathcal{A}\}$ and $\{i : D_s^t(i) \in \mathcal{B}\}$ should not be empty. Hence, we consider the following two cases separately:

- **Case I:** $D_s^t(1) \in \mathcal{A}$ and $D_s^t(1) \in \mathcal{B}$.
- **Case II:** $D_s^t(1) \notin \mathcal{A}$ or $D_s^t(1) \notin \mathcal{B}$.

As $D_s^t(1)$ is the shift minimal walk in $\dot{\mathcal{F}}^t \cap \mathcal{F}_s^t$ for $s = 0$ and 1, and as the subsets $\dot{\mathcal{A}}$ and $\dot{\mathcal{B}}$ are non-empty, we have that Case I holds if $s = 0$ or 1. So in Case II we may assume that $s \ge 2$.

- **Case I:** *Suppose that* $D_s^t(1) \in \mathcal{A}$ *and* $D_s^t(1) \in \mathcal{B}$.
  First, we suppose that $I = J = i_{\max}$. Since $D_s^t(i_{\max}) \in \mathcal{A}$, Fact 20 gives that

$$\text{dual}_t\left(D_s^t(i_{\max})\right) = [n] \setminus \{t+s, t+s+1, \ldots, t+2s\}$$

is not contained in $\mathcal{B}$. Consequently, Fact 19 gives that each walk $B \in \mathcal{B}$ satisfies $B \not\to \text{dual}_t\left(D_s^t(i_{\max})\right)$. Hence, $\mathcal{B} \subset \mathcal{F}_s^t$ holds. Similarly, $J = i_{\max}$ implies $\mathcal{A} \subset \mathcal{F}_s^t$. Therefore, we have

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \le \left(\mu_p(\mathcal{F}_s^t)\right)^2$$

with equality holding iff $\mathcal{A} = \mathcal{B} = \mathcal{F}_s^t$. This together with $1/(t+1) \le p \le 2/(t+3)$ implies (22) and (27).

Therefore, we can assume that $I \ne i_{\max}$ or $J \ne i_{\max}$. Without loss of generality, let $I \ne i_{\max}$. The following holds for every $s \ge 0$ (not only for $0 \le s \le 2$).

Claim 28. *If* $I \ne i_{\max}$ *and* $0 \le s < \infty$, *then*

$$\mu_p(\mathcal{F}_s^t \setminus \mathcal{A}) \ge \binom{t+s-1}{s} p^{t+s} q^{s+I+1}(1-\alpha) \tag{23}$$

and

$$\mu_p(\mathcal{B} \setminus \mathcal{F}_s^t) \le \alpha^{t+I}. \tag{24}$$

Proof. First, we show (23). Consider a walk W that hits $(s, t+s)$ and satisfies $W \to D_s^t(I+1)$. Since $D_s^t(I+1) \notin \mathcal{A}$, Fact 19 gives $W \in \mathcal{F}_s^t \setminus \mathcal{A}$. Also, W must hit $Q_1 = (s, t-1)$ and $Q_2 = (s+I+1, t+s)$. The number of walks from $(0, 0)$ to Q_1

is $\binom{t+s-1}{s}$, then there is the unique walk from Q_1 to Q_2 which hits $(s, t+s)$. So the weight of the family of all such walks is $\binom{t+s-1}{s}p^{t+s}q^{s+I+1}$. To satisfy $W \to D_{I+1}$, the walk W must not hit the line $y = x + (t - I)$. By Lemma 2(i), this happens with probability at least $1 - \alpha$, which yields (23).

Next, we show (24). Since $D_s^t(I) \in \mathcal{A}$, we have that $\mathrm{dual}_t\left(D_s^t(I)\right) \notin \mathcal{B}$, and hence, each walk $B \in \mathcal{B}$ must hit $(0, t+s), (1, t+s), \ldots, (s, t+s)$, or $y = x + (t + I)$. Note that each walk hitting $(0, t+s), (1, t+s), \ldots,$ or $(s, t+s)$ is contained in $\mathcal{F}_s^t$. Thus, each walk $B \in \mathcal{B} \setminus \mathcal{F}_s^t$ hits $y = x + (t + I)$. Lemma 2(i) gives (24). $\square$

Claim 28 together with $0 \le s \le 2$ implies the following.

Corollary 29. *If $I \ne i_{\max}$ and $0 \le s \le 2$, then $\mu_p(\mathcal{B} \setminus \mathcal{F}_s^t) < 0.99\mu_p(\mathcal{F}_s^t \setminus \mathcal{A})$.*

Proof. Inequalities (23) and (24) give that

$$\frac{\mu_p(\mathcal{F}_s^t \setminus \mathcal{A})}{\mu_p(\mathcal{B} \setminus \mathcal{F}_s^t)} \ge \frac{\binom{t+s-1}{s}p^{t+s}q^{s+I+1}(1-\alpha)}{\alpha^{t+I}} = \binom{t+s-1}{s}p^s q^{t+s+1}\left(\frac{q^2}{p}\right)^I(1-\alpha)$$
$$\ge \binom{t+s-1}{s}p^s q^{t+s}\left(\frac{q^2}{p}\right)(q-p) = \binom{t+s-1}{s}p^{s-1}q^{t+s+2}(q-p),$$

where the second inequality holds since $q^2/p > 1$ for $p < 0.38$. Since $1/(t+1) \le p \le 2/(t+3)$, one can easily check that

$$\binom{t+s-1}{s}p^{s-1}q^{t+s+2}(q-p) > 1.02$$

if $s = 0$ and $t \ge 17$, or if $s = 1$ and $t \ge 12$, or if $s = 2$ and $t \ge 22$. $\square$

On the other hand, we also claim that

$$\mu_p(\mathcal{A} \setminus \mathcal{F}_s^t) < 0.99\mu_p(\mathcal{F}_s^t \setminus \mathcal{B}). \tag{25}$$

Indeed, if $J \ne i_{\max}$, then Corollary 29 gives (25). Otherwise, $J = i_{\max}$, and hence, $\mathcal{A} \subset \mathcal{F}_s^t$. Thus, (25) trivially holds.

We infer that

$$\begin{aligned}\mu_p(\mathcal{A}) + \mu_p(\mathcal{B}) &= \left(\mu_p(\mathcal{A} \cap \mathcal{F}_s^t) + \mu_p(\mathcal{A} \setminus \mathcal{F}_s^t)\right) + \left(\mu_p(\mathcal{B} \cap \mathcal{F}_s^t) + \mu_p(\mathcal{B} \setminus \mathcal{F}_s^t)\right)\\ &< \left(\mu_p(\mathcal{A} \cap \mathcal{F}_s^t) + 0.99\mu_p(\mathcal{F}_s^t \setminus \mathcal{A})\right) + \left(\mu_p(\mathcal{B} \cap \mathcal{F}_s^t) + 0.99\mu_p(\mathcal{F}_s^t \setminus \mathcal{B})\right)\\ &< 2\mu_p(\mathcal{F}_s^t),\end{aligned}$$

where the inequality follows from Corollary 29 and (25). Therefore,

$$\sqrt{\mu_p(\mathcal{A})\mu_p(\mathcal{B})} \le \frac{\mu_p(\mathcal{A}) + \mu_p(\mathcal{B})}{2} < \mu_p(\mathcal{F}_s^t),$$

which gives (22) without equality.

- **Case II:** *Suppose that $D_s^t(1) \notin \mathcal{A}$ or $D_s^t(1) \notin \mathcal{B}$.*

As we observed before Case I, in Case II we may assume that $s \ge 2$. Also, without loss of generality, we let $D_s^t(1) \notin \mathcal{A}$, so every $A \in \mathcal{A}$ satisfies $A \not\to D_s^t(1)$.

For $1 \le i \le n - t - 5$, let

$$E(i) := [t-1] \cup \{t+1, t+3, t+4\} \cup \{t+4+i+2j \in [n] : j = 1, 2, \ldots\}.$$

For any $A \in \dot{\mathcal{A}} \ne \emptyset$, we have $A \to E(1)$, and hence, Fact 18 gives that $E(1) \in \mathcal{A}$. Since $\{i : E(i) \in \mathcal{A}\} \ne \emptyset$, the number

$$K := \max\{i : E(i) \in \mathcal{A}\}$$

is well-defined.

Let $A \in \dot{\mathcal{A}}$. The walk A must hit $(2, t+2)$ without hitting $(0, t)$ or $(1, t+1)$. Also, since $D_s^t(1) \notin \mathcal{A}$, the walk A must hit $(1, t)$. The weight of the family of all such walks is $tp^{t+2}q^2$. From $(2, t+2)$, the walk A moves to the right and hits $(3, t+2)$. Then it must not hit the line $y = x + t$. Lemma 3 implies that this happens with probability less than $q(1 - \alpha + \epsilon)$ where $\epsilon \to 0$ as n tends to ∞. Let n be sufficiently large that $\epsilon \le \alpha$. Then, we have that

$$\mu_p(\dot{\mathcal{A}}) < tp^{t+2}q^3(1 - \alpha + \epsilon) \le tp^{t+2}q^3.$$

For $\ddot{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ we use the trivial bounds $\mu_p(\ddot{\mathcal{A}}) \le \alpha^{t+1}$ and $\mu_p(\tilde{\mathcal{A}}) \le \alpha^{t+1}$ from Lemma 2. Consequently we have

$$\mu_p(\mathcal{A}) = \mu_p(\dot{\mathcal{A}}) + \mu_p(\ddot{\mathcal{A}}) + \mu_p(\tilde{\mathcal{A}}) < tp^{t+2}q^3 + 2\alpha^{t+1}. \tag{26}$$

On the other hand,

$$\begin{aligned}\mathrm{dual}_t(E(K)) = [t] &\cup \{t+2\} \cup \{t+5, \ldots, t+5+K\}\\ &\cup \{t+5+2j \in [n] : j = 1, 2, \ldots\}.\end{aligned}$$

Since $\mathrm{dual}_t(E(K)) \notin \mathcal{B}$, every walk $B \in \mathcal{B}$ must hit one of $(0, t+1)$, $(1, t+2)$, $(2, t+2)$, or the line $y = x + t + K$. Thus we have

$$\begin{aligned}\mu_p(\mathcal{B}) &\le p^{t+1} + (t+1)p^{t+2}q + \left((t+1) + \binom{t+2}{2}\right) p^{t+2}q^2 + \alpha^{t+K}\\ &\le \left(1 + (t+1)pq + \frac{(t+1)(t+4)}{2}pq^2\right) p^{t+1} + \alpha^{t+1}. \qquad (27)\end{aligned}$$

Inequalities (26) and (27) give that

$$\begin{aligned}\frac{\mu_p(\mathcal{A})\mu_p(\mathcal{B})}{\left(\mu_p(\mathcal{F}_1^t)\right)^2} &\le \frac{\left(tp^{t+2}q^3 + 2\alpha^{t+1}\right)\left(1 + (t+1)pq + \frac{(t+1)(t+4)}{2}pq^2 + 1/q^{t+1}\right)p^{t+1}}{\left((t+2)p^{t+1}q\right)^2}\\ &\le \frac{\left(tpq^3 + \frac{2e^2}{q}\right)\left(1 + (t+1)pq + \frac{(t+1)(t+4)}{2}pq^2 + \frac{e^2}{q}\right)}{(t+2)^2q^2}, \qquad (28)\end{aligned}$$

where the second inequality follows from (4). Since pq, pq^2, pq^3 and $1/q$ are increasing in p for $p \le 0.2$, expression (28) is maximized when $p = 2/(t+3)$. One can check that (28) with $p = 2/(t+3)$ is at most 0.99 for $t \ge 180$. Therefore, for $t \ge 180$,

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) < 0.99\left(\mu_p(\mathcal{F}_1^t)\right)^2, \qquad (29)$$

which completes our proof of Lemma 27. $\square$

We have proved the inequality (3) under Assumption 7. The uniqueness of the optimal families in Theorem 1 now follows from Lemma 6. This completes the proof of Theorem 1.

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Appendix A. Omitted calculations

A.1. Calculations for Claim 16

We give here the calculations for the cases $(s, s') = (3, 3)$, $(3, 2)$ and $(2, 2)$, omitted from the proof of Claim 16.

Case: $(s, s') = (3, 3)$

Noting that $u = v = t$ we get that

$$f(u, 3, p) = f(v, 3, p) = \alpha^{t+1} + \binom{t+6}{3}\frac{t+1}{t+4}p^{t+3}q^3(1-\alpha) < e^2\frac{p^{t+1}}{q} + \frac{(t+6)(t+5)(t+1)}{6}p^{t+3}q^3.$$

Thus as $\mu_p(\mathcal{F}_1^t) > (t+2)p^{t+1}q$ we get that

$$\begin{aligned}\frac{f(u, 3, p)}{\mu_p(\mathcal{F}_1^t)} &< \frac{e^2}{q^2(t+2)} + \frac{p^2q^2(t+6)(t+5)(t+1)}{6(t+2)}\\ &< \frac{2e^2}{t+2} + \frac{4(t+6)(t+5)(t+1)}{6(t+2)(t+3)^2}.\end{aligned}$$

This is less than .99 for $t \ge 52$, its square $\frac{g(3,3)}{(\mu_p(\mathcal{F}_1^t))^2}$ is also.

Case: $(s, s') = (3, 2)$

Noting that $u = t - 1$ and $v = t + 1$ we get that

$$f(u, 3, p) = \alpha^t + \binom{t+5}{3} \frac{t}{t+3} p^{t+2} q^3 (1-\alpha) < e^2 p^t + \frac{(t+5)(t+4)t}{6} p^{t+2} q^3,$$

and we get

$$f(v, 2, p) = \alpha^{t+2} + \binom{t+5}{2} \frac{t+2}{t+4} p^{t+3} q^2 (1-\alpha) < \frac{e^2 p^{t+2}}{q^2} + \frac{(t+5)(t+2)}{2} p^{t+3} q^2.$$

Thus as $(\mu_p(\mathcal{F}_1^t))^2 > (t+2)^2 p^{2t+2} q^2$ we get that

$$\begin{aligned} \frac{g(3,2)}{(\mu_p(\mathcal{F}_1^t))^2} &< \frac{e^4}{q^4(t+2)^2} + \frac{e^2 p(t+5)}{2(t+2)} + \frac{e^2 p^2 (t+5)(t+4)t}{6q(t+2)^2} + \frac{p^3 q^3 (t+5)^2 (t+4)t}{12(t+2)} \\ &< \frac{2e^4}{(t+2)^2} + \frac{e^2(t+5)}{(t+2)(t+3)} + \frac{4e^2(t+5)(t+4)t}{6(t+2)^2(t+3)(t+1)} + \frac{2(t+5)^2(t+4)t}{3(t+2)(t+3)^3}. \end{aligned}$$

This is less than .99 for $t \geq 51$.

Case: $(s, s') = (2, 0)$

Noting that $u = t - 2$ and $v = t + 2$ we get that

$$f(u, 2, p) = \alpha^{t-1} + \binom{t+2}{2} \frac{t-1}{t+1} p^t q^2 (1-\alpha) < e^2 \frac{p^{t-1}}{q} + \frac{(t+2)(t-1)}{2} p^t q^2,$$

and we get

$$f(v, 0, p) = \alpha^{t+3} + p^{t+2} q(1-\alpha) < \frac{e^2 p^{t+3}}{q^3} + p^{t+2} q.$$

Thus as $(\mu_p(\mathcal{F}_1^t))^2 > (t+2)^2 p^{2t+2} q^2$ we get that

$$\begin{aligned} \frac{g(2,0)}{(\mu_p(\mathcal{F}_1^t))^2} &< \frac{e^4}{q^6(t+2)^2} + \frac{e^2}{(t+2)^2 pq^2} + \frac{e^2 p(t-1)}{2q^3(t+2)} + \frac{(t-1)q}{2(t+2)} \\ &< \frac{2e^4}{(t+2)^2} + \frac{e^2(t+1)^3}{(t+2)^2 t^2} + \frac{2e^2(t-1)}{(t+3)(t+2)} + \frac{(t-1)(t+1)}{2(t+2)(t+3)}. \end{aligned}$$

This is less than .99 for $t \geq 42$.

A.2. Proofs of Claims 23, 25 and 26

Proof of Claim 23. Let $\epsilon > 0$ be given and let $\delta = \epsilon / b_1(p, t)$. As $s' = 1$ we have $\mu_p(\mathcal{B}) = \mu_p(\tilde{\mathcal{B}}) + \mu_p(\mathcal{B}_1)$. Noting that $D_2^{t-1}(I) \in \mathcal{A}$ and

$$\mathrm{dual}_t(D_2^{t-1}(I)) = [t+1] \cup [t+4, t+4+I] \cup \{t+6+I, t+8+I, \ldots\} \notin \mathcal{B}$$

we see that every walk in $\mathcal{B}$ must hit at least one of $(0, t+2)$, $(1, t+2)$, and the line $L : y = x + (t+1+I)$. Also all walks in $\tilde{\mathcal{B}} \subset \mathcal{F}^v = \mathcal{F}^{t+1}$ hit the line $L' : y = x + (t+2)$. Thus we get

$$\begin{aligned} \mu_p(\tilde{\mathcal{B}}) &\leq \mu_p(\text{walks in } \tilde{\mathcal{B}} \text{ hitting } L) + \mu_p(\text{walks in } \tilde{\mathcal{B}} \text{ not hitting } L \text{ but hitting } (0, t+2)) \\ &\quad + \mu_p(\text{walks in } \tilde{\mathcal{B}} \text{ not hitting } L \text{ or } (0, t+2) \text{ but hitting } (1, t+2) \text{ and } L') \\ &\leq \alpha^{t+1+I} + p^{t+2}(1 - \alpha^{I-1} + \delta) + (t+2) p^{t+2} q(\alpha - \alpha^I + \delta/2). \end{aligned} \tag{A.1}$$

On the other hand, as $D_1^{t+1}(J+1) \notin \mathcal{B}$, we have that $\mathcal{B}_1 \subset \bar{\mathcal{F}}_1^{t+1} \setminus \mathcal{W}$ where

$$\mathcal{W} = \{W \in \dot{\mathcal{F}}^{t+1} \cap \mathcal{F}_1^{t+1} : W \to D_1^{t+1}(J+1)\}.$$

Walks in $\bar{\mathcal{F}}_1^{t+1}$ hit $(1, t+2)$ without hitting $(0, t+2)$ then do not go above $y = x + t + 1$ so we have that

$$\mu_p(\bar{\mathcal{F}}_1^{t+1}) \leq (t+2) p^{t+2} q(1 - \alpha + \delta/2). \tag{A.2}$$

Walks in $\mathcal{W}$ are those in $\bar{\mathcal{F}}_1^{t+1}$ that after hitting $(1, t+2)$ go over to $(J+2, t+2)$, which is on the line $y = x + t - J$, and then never cross this line. So

$$\mu_p(\mathcal{W}) \geq (t+1)p^{t+2}q^{J+2}(1-\alpha). \tag{A.3}$$

Combining (A.1)–(A.3) yields that

$$\mu_p(\mathcal{B})/p^{t+2} \leq q^{-(t+2)}\alpha^{I-1} + (1-\alpha^{I-1}+\delta) + (t+2)q(1-\alpha^I+\delta) - (t+1)q^{J+2}(1-\alpha).$$

Rearranging we get

$$\mu_p(\mathcal{B})/p^{t+2} \leq (1+\delta)b_1(p,t) + b_2(p,t)\alpha^{I-1} - b_3(p,t)q^{J-2},$$

as needed.

Since $\frac{\partial}{\partial p}b_2(p,t) = (t+2)(q^{-(t+3)}-1) > 0$ it follows that $b_2(p,t) \leq b_2(\frac{2}{t+3}, t) = (\frac{t+3}{t+1})^{t+2} - 1 - \frac{2(t+2)}{t+3}$, which is decreasing in t, so for $t \geq 18$, $b_2(p,t) \leq b_2(\frac{2}{t+3}, t) \leq b_2(\frac{2}{21}, 18) < 4.5$.

As $q^3(1-\alpha)$ is decreasing in p, we get, by evaluating it at $p = 2/(t+3)$, that $b_3(p,t) \geq q\frac{(t-1)(t+1)^3}{(t+3)^3} > (t-7)q$. □

Proof of Claim 25. Let ϵ be given and let $\delta = \epsilon/(tq)$. We use that $\mathcal{A} = \tilde{\mathcal{A}} \cup \mathcal{A}_1$. We have by arguments similar to in the proof of Claim 22, or from the inequalities (11) and (12) of [7], that

$$\mu_p(\tilde{\mathcal{A}}) \leq \alpha^{t+J-1} + p^t + tp^t q\alpha,$$
$$\mu_p(\mathcal{A}_1) \leq \mu_p(\bar{\mathcal{F}}_1^{t-1}) < tp^t q(1-\alpha+\delta),$$

where the second inequality follows by choosing n sufficiently large.

We also use that $\mathcal{A}_1 \subset \bar{\mathcal{F}}_1^{t-1} \setminus \mathcal{W}$, where

$$\mathcal{W} = \{W \in \dot{\mathcal{F}}^{t-1} \cap \mathcal{F}_1^{t-1} : W \to D_1^{t-1}(I+1)\}.$$

A path $W \in \mathcal{W}$ hits $(1, t)$ without hitting $(0, t)$ and then goes over to $(I+2, t)$ on the line $y = x + (t-I-2)$, and afterwards never crosses this line. So

$$\mu_p(\mathcal{W}) > (t-1)p^t q^{I+2}(1-\alpha).$$

Together, this gives

$$\begin{aligned}\mu_p(\mathcal{A}) &< \alpha^{t+J-1} + p^t + tp^t q(1+\delta) - (t-1)p^t q^{I+2}(1-\alpha) \\ &< p^t\left((1+tq) + \frac{\alpha^{J-1}}{q^t} - (t-1)q^{I+2}(1-\alpha) + tq\delta\right),\end{aligned}$$

which yields the main inequality of the claim.

That $q^{-t} < e^2 < 7.4$ was observed in the proof of Claim 22 and that $(t-1)q^3(1-\alpha) > (t-7)q$ can be shown as in the proof of Claim 23. □

Proof of Claim 26. Let ϵ be given and let $\delta = p\epsilon$. We use that $\mathcal{B} = \tilde{\mathcal{B}} \cup \mathcal{B}_0 \subset \tilde{\mathcal{B}} \cup (\bar{\mathcal{F}}_0^{t+1} \setminus \mathcal{W})$, where

$$\mathcal{W} = \{W \in \dot{\mathcal{F}}^{t+1} \cap \mathcal{F}_0^{t+1} : W \to D_0^{t+1}(J+1)\}.$$

From the inequalities (14) and (15) of [7] we have that

$$\mu_p(\tilde{\mathcal{B}}) + \mu_p(\bar{\mathcal{F}}_0^{t+1}) \leq \alpha^{t+1+I} + p^{t+1}(1+\delta).$$

A path W in $\mathcal{W}$ hits $(0, t+1)$ goes over to $(J+1, t+1)$ on the line $y = x + (t-J)$, and then never crosses this line, so

$$\mu_p(\mathcal{W}) \geq p^{t+1}q^{J+1}(1-\alpha).$$

Consequently it follows that

$$\mu_p(\mathcal{B}) \leq p^{t+1} + \alpha^{t+1+I} - p^{t+1}q^{J+1}(1-\alpha) + p^{t+1}\delta,$$

which yields the main inequality of the claim. The bound for $b_2(p,t)$ was shown in the proof of the previous claim, and the bound for $b_3(p,t)$ can be verified for $t \geq 20$. □

A.3. Other computations

We verify that y/z is decreasing for Case 1 of Claim 23. Recall that $z = t + 2 - (t + 1)p = (t + 2)q + p$ and that $y = 1 + (t+2)q + 4.5\alpha^2 = z + (4.5\alpha^2 + q)$, so $y/z = 1 + \frac{r}{z}$, where $r = 4.5\alpha^2 + q$. We show this is increasing by showing that $zr' - rz' > 0$, where r' and z' denote derivatives with respect to p. Noting that $\frac{\partial}{\partial p} q = -1$ and $\frac{\partial}{\partial p}\alpha = q^{-2}$ we compute

$$\begin{aligned} zr' - rz' &= z(9\alpha q^{-2} - 1) - r(-(t+1)) \\ &= 4.5(t+1)\alpha^2 + 9(t+2)\frac{p}{q^2} + 9\frac{p^2}{q^3} - 1 \\ &> 9(t+2)\frac{p}{q^2} - 1 > 9(t+2)\frac{t+1}{t^2} - 1 > 8, \end{aligned}$$

as needed. In the last line we use $p = \frac{1}{t+1}$ as p/q^2 is increasing in p.

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